PMM U.S.S.R.,Vol.52,No.3,pp.377-382,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

A METHOD OF SYMMETRIZING FUNCTIONS AND ITS APPLICATION TO CERTAIN PROBLEMS IN ELASTICITY THEORY FOR NON-UNIFORM BODIES*

S.R. BRUDNYI and E.I. SHIFRIN

A symmetrization operation is introduced for functions defined in bounded domains and vanishing on the boundary. The properties of the operation introduced are studied and its connection with Schwarz symmetrization is analysed. Examples are considered of the application of the apparatus developed for constructing isoperimetric estimates in problems of the torsion and longitudinal vibrations of an inhomogeneous rod. The stiffness estimate obtained in the problem of the torsion of a non-uniform rod is a generalization of the Polya isoperimetric inequality known in the theory of elasticity for the stiffness of a uniform rod under torsion.

The solution of many problems in the theory of elasticity encounters serious mathematical difficulties. Nevertheless, it is not so much the stress and displacement fields that are often of practical interest as are certain of their integral characteristics (for instance, the stiffness of an elastic rod under torsion, the frequency of the fundamental of the natural vibrations of a membrane, the first critical force of a compressed rod, etc.). In a number of cases they have been successfully estimated without finding the complete solution of the problem. Among all the possible estimates, the most effective ones are the isoperimetric estimates in which the desired quantity is estimated in terms of the appropriate characteristic of the solution of the simpler problem that allows an analytic or effective numerical solution.

The construction of isoperimetric inequalities for the solutions of boundary value problems is based, as a rule, on the application of Steiner or Schwarz symmetrization operations for function level lines /1/ which retain the L_p-norm of the functions and do not magnify the corresponding norm of its gradient. This apparatus turns out to be effective for constructing estimates of solutions of a certain class of differential equations with constant coefficients /1/ and variable coefficients in the smallest terms /2, 3/. Utilization of Steiner and Schwarz symmetrization also enables isoperimetric inequalities to be obtained for a certain type of pseudodifferential equation /4/. However, the methods developed in /1-4/ do not enable estimates to be obtained for solutions of boundary value problems for differential equations with variable coefficients in the highest derivatives, and it is such equations that are encountered in elasticity theory problems for non-uniform bodies. The reason for the difficulties occurring here is that the L_p-norm must be estimated for the gradient containing weight functions. Standard symmetrization operations of function level lines transform its gradient in an arbitrary manner, whereupon effective reconstruction of the weight function in the symmetrized domain is not successful and, therefore, neither is the required estimate.

Below we propose a new symmetrization operation that enables the above-mentioned difficulty to be overcome and examples of its application are examined.

1. *Definition.* Let a function $f(\mathbf{x})$ be defined in a bounded domain $G \in R^n$, possess a summable modulus of the gradient in this domain, and vanish on its boundary ∂G. Let K be a sphere whose volume equals the volume of G while the centre is at the origin, $g(\mathbf{x}) = g(r)$ is a function defined in K, equally measurable with $|\nabla f(\mathbf{x})|$ and no decreasing as the radius $r = |\mathbf{x}|$ increases. We define the function

*Prikl.Matem.Mekhan.,52,3,486-492,1988

$$F(r)=\int_r^R g(t)\,dt$$

in the sphere K, where R is the radius of the sphere K. We will call the operation of setting the function $f(\mathbf{x})$ in correspondence with the function $F(r)$ the SE-symmetrization of the function f, and use the notation $SE(f)$.

We recall that the functions $A(\mathbf{x})$ and $B(\mathbf{x})$ are called equally measurable if $\forall a, b$ measures of the sets $\{\mathbf{x}\colon a < A(\mathbf{x}) < b\}$ and $\{\mathbf{x}\colon a < B(\mathbf{x}) < b\}$ are equal.

We will examine the properties of the function $SE(f)$ constructed. It is spherically-symmetric $SE(f)(x) = SE(f)(r)$, where $SE(f)(R) = 0$, i.e., $SE(f)$ vanishes on the boundary of the sphere K. Moreover

$$|\nabla SE(f)(\mathbf{x})| = |dSE(f)(r)/dr| = g(r)$$

Therefore, the functions $|\nabla SE(f)(\mathbf{x})|$ and $|\nabla f(\mathbf{x})|$ are equally measurable.

Theorem 1.

$$\int_K SE(f)(\mathbf{x})\,d\mathbf{x} \geqslant \int_G |f(\mathbf{x})|\,d\mathbf{x} \tag{1.1}$$

Proof. We first note the following fact. Let $A(r)$ be a spherically-symmetric function defined in a sphere K of radius R, where $A(R) = 0$ and $A(r)$ does not increase as r increases. Then the following representation holds:

$$V_A=\int_K A(r)\,d\mathbf{x}=c_n\int_0^R r^{n-1}A(r)\,dr=n^{-1}\int_0^R J_A(r)\,dr \tag{1.2}$$

$$J_A(r)=c_n\int_r^R t^{n-1}|A'(t)|\,dt,\quad c_n=ns_n \tag{1.3}$$

(s_n is the volume of the unit sphere in R^n). The validity of (1.2) can be proved by integration by parts. Expression (1.3) is obviously equivalent to the following:

$$J_A(\rho)=\int_{L_\rho}|\nabla A(\mathbf{x})|\,d\mathbf{x},\quad L_\rho=\{\mathbf{x}\in K:\rho<r<R\} \tag{1.4}$$

Let $S(f)(\mathbf{x}) = S(f)(r)$ denote a function defined in K and the result of application of the Schwarz symmetrization operation to the function $|f(\mathbf{x})|$ /1/. As is well-known /1/

$$\int_K S(f)(\mathbf{x})\,d\mathbf{x}=\int_G |f(\mathbf{x})|\,d\mathbf{x} \tag{1.5}$$

It follows from the properties of the function $SE(f)$ considered above and the known properties of the function $S(f)$ /1/ that the representation (1.2) is valid.

Let $S(f)(\rho) = \varkappa$. Since $S(f)(r)$ is a non-increasing function of the radius r, the set $\Omega = \{\mathbf{x} \in K\colon S(f) \leqslant \varkappa\}$ is identical with the ring L_ρ. Hence, taking (1.4) and the properties of the Schwarz symmetrization operation /1/ into account, we have

$$J_{S(f)}(\rho)=\int_\Omega |\nabla S(f)|\,d\mathbf{x}\leqslant\int_\omega |\nabla f(\mathbf{x})|\,d\mathbf{x} \tag{1.6}$$

$$\omega = \{\mathbf{x} \in G\colon |f(\mathbf{x})| \leqslant \varkappa\}$$

Note that the measures of the sets Ω and ω are equal. Since the functions $|\nabla SE(f)(\mathbf{x})|$ and $|\nabla f(\mathbf{x})|$ are equally measurable by construction and $|\nabla SE(f)(\mathbf{x})|$ does not decrease as r increases, we have

$$J_{SE(f)}(\rho)=\int_\Omega |\nabla SE(f)(r)|\,d\mathbf{x}\geqslant\int_\omega |\nabla f(\mathbf{x})|\,d\mathbf{x} \tag{1.7}$$

It follows from (1.6) and (1.7) that

$$J_{SE(f)}(r) \geqslant J_{S(f)}(r),\quad \forall r \in [0, R] \tag{1.8}$$

Hence, taking account of the representation (1.2) for the functions $SE(f)$ and $S(f)$ and using (1.5) we obtain the inequality (1.1).

Remark 1. It does not follow from the theorem proved that $SE(f)(r) \geqslant S(f)(r)$ and this inequality is not satisfied in the general case. However, in the one-dimensional case $n = 1$

it does hold. In fact, for $n = 1$ it follows from the definition of the functions $J_{SE(f)}$ and $J_{S(f)}$ that they simply agree with $2SE(f)(r)$ and $2S(f)(r)$, respectively. Consequently, by virtue of (1.8)

$$SE(f)(r) \geqslant S(f)(r), \quad \forall r \in [0, R], \quad n = 1 \tag{1.9}$$

The following indicates the possibility of violating inequality (1.9) for $n \geqslant 2$

Lemma. Let $f(\mathbf{x}) = f(r)$ be a non-negative, monotonically decreasing function of r defined in a sphere of radius R and vanishing on its boundary. Then for $n \geqslant 2$

$$S(f)(0) \geqslant SE(f)(0) \tag{1.10}$$

Proof. Obviously $S(f)(r) = f(r)$ and

$$S(f)(0) = \int_0^R \left| \frac{dS(f)(r)}{dr} \right| dr \tag{1.11}$$

On the basis of (1.3)

$$|dS(f)(r)/dr| = -(c_n^{-1} r^{-n+1})\, dJ_{S(f)}(r)/dr \tag{1.12}$$

Substituting (1.12) into (1.11) and integrating by parts, we obtain

$$S(f)(0) = c_n^{-1}\left[(n-1)\int_0^R \frac{J_{S(f)}(0) - J_{S(f)}(t)}{t^n}\, dt + \frac{J_{S(f)}(0)}{R^{n-1}}\right] \tag{1.13}$$

We similarly obtain

$$SE(f)(0) = c_n^{-1}\left[(n-1)\int_0^R \frac{J_{SE(f)}(0) - J_{SE(f)}(t)}{t^n}\, dt + \frac{J_{SE(f)}(0)}{R^{n-1}}\right] \tag{1.14}$$

Since $J_{SE(f)}(0) = J_{S(f)}(0)$ by construction and the inequality (1.8) holds, inequality (1.10) follows from (1.13) and (1.14).

If the function $f(r)$ under consideration is not invariant under SE-symmetrization, then in a certain set $\delta \subset [0, R]$ $J_{SE(f)}(r) > J_{S(f)}(r)$ and therefore, the strict inequality $S(f)(0) > SE(f)(0)$ is satisfied.

Remark 2. Let a non-negative function f that vanishes on ∂G be defined in the domain $G \in R^n$. This function yields a certain surface in the space R^{n+1}. Consider the body $T_1 \in R^{n+1}$ formed by the surface f and the domain G. Its volume V_1 and the surface area S_1 are found from the formulas

$$V_1 = \int_G f(\mathbf{x})\, d\mathbf{x}, \quad S_1 = \int_G [1 + \sqrt{1 + |\nabla f|^2}]\, d\mathbf{x}$$

We apply SE-symmetrization to the function $f(\mathbf{x})$. Consequently, we obtain an axisymmetric body $T_2 \in R^{n+1}$ formed by the surface $SE(f)$ and the sphere K, whose volume V_2 and surface area S_2 are expressed in the form

$$V_2 = \int_K SE(f)(\mathbf{x})\, d\mathbf{x}, \quad S_2 = \int_K [1 + \sqrt{1 + |\nabla SE(f)|^2}]\, d\mathbf{x}$$

It follows from the properties of SE-symmetrization examined above that $V_2 \geqslant V_1$, $S_2 = S_1$. Therefore, SE-symmetrization of the body T_1 (in the above-mentioned sense) conserves the surface area and does not reduce its volume.

It is interesting to compare the body T_2 and the body T_3 formed by the surface $S(f)$ and the sphere K. According to the well-known properties of the Schwarz symmetrization operation /1/, the volume V_3 and the surface are S_3 of the body T_3 are related to the corresponding characteristics of the body T_1 as follows: $V_3 = V_1$, $S_3 \leqslant S_1$. In other words, the volume is conserved and the surface area is not increased when constructing the body T_3 by using Schwarz symmetrization.

By virtue of (1.9), the relation $T_3 \subset T_2$ holds in the case $n = 1$.

2. We consider certain examples illustrating the possibility of applying SE-symmetrization to obtain isoperimetric estimates.

Pure torsion of an inhomogeneous rod of simply-connected section. This problem reduces to solving the equation /5/ ($\mu(\mathbf{x})$ is the shear modulus, $G \in R^2$ is a simply-connected domain)

$$(\nu\Phi_{,1})_{,1} + (\nu\Phi_{,2})_{,2} = -2, \quad \mathbf{x} \in G, \quad \Phi\,|_{\partial G} = 0$$
$$\nu(\mathbf{x}) = \mu^{-1}(\mathbf{x}), \quad 0 < \mu(\mathbf{x}) < \infty \tag{2.1}$$

The rod stiffness under torsion is expressed by the quantity /5/

$$C = 2\int_G \Phi(\mathbf{x})\, d\mathbf{x}$$

Theorem 2. The stiffness under torsion of a non-uniform rod of simply-connected section does not exceed the stiffness of a circular rod of the same sectional area with a shear modulus equally measurable with the original modulus, axisymmetric and non-decreasing as the radius increases.

(This theorem generalizes the isoperimetric theorem of the theory of the torsion of uniform rods /1/ in a natural manner).

Proof. The solution of problem (2.1) yields a maximum of the functional

$$U = \sup_{\varphi,\ \varphi|_{\partial G}=0} \left[4\int_G \varphi\, d\mathbf{x} - \int_G \nu\,|\nabla\varphi|^2\, d\mathbf{x}\right] \tag{2.2}$$

where $U_{\max} = C$ /5/. Let $\nu_0(\mathbf{x})$ be a function equally measurable with the function $\nu(\mathbf{x})$ and oppositely directed to the function $|\nabla\Phi(\mathbf{x})|$, where $\Phi(\mathbf{x})$ is the solution of problem (2.1). We recall that the functions $A(\mathbf{x})$ and $B(\mathbf{x})$ are called codirectional if

$$F(\mathbf{x}_1, \mathbf{x}_2) = [A(\mathbf{x}_1) - A(\mathbf{x}_2)]\,[B(\mathbf{x}_1) - B(\mathbf{x}_2)] \geqslant 0, \quad \forall \mathbf{x}_1, \mathbf{x}_2 \tag{2.3}$$

and oppositely directed if $F(\mathbf{x}_1, \mathbf{x}_2) \leqslant 0$ /1/.

Since the function $\nu_0(\mathbf{x})$ is oppositely directed to $|\nabla\Phi(\mathbf{x})|$, then in the general case it can be represented in the form $\nu_0(\mathbf{x}) = \nu_1(|\nabla\Phi(\mathbf{x})|)$ /1/.

Later we shall use the following property of codirected and oppositely directed functions /1/. Let the functions f and g be defined and non-negative in the domain $\Omega \in R^n$. Then

$$\int_\Omega f_- g_-\, d\mathbf{x} \leqslant \int_\Omega fg\, d\mathbf{x} \leqslant \int_\Omega f_+ g_+\, d\mathbf{x}$$

where $f_\pm$ and $g_\pm$ are equally measurable with f and g, respectively, on Ω and f_+ is codirectional with g_+ while f_- and g_- are oppositely directed.

On the basis of this property

$$\int_G \nu(\mathbf{x})\,|\nabla\Phi(\mathbf{x})|^2\, d\mathbf{x} \geqslant \int_G \nu_0(\mathbf{x})\,|\nabla\Phi(\mathbf{x})|^2\, d\mathbf{x}$$

Hence and from (2.2) it follows that

$$C = 4\int_G \Phi\, d\mathbf{x} - \int_G \nu(\mathbf{x})\,|\nabla\Phi(\mathbf{x})|^2\, d\mathbf{x} \leqslant 4\int_G \Phi\, d\mathbf{x} - \int_G \nu_0(\mathbf{x})\,|\nabla\Phi(\mathbf{x})|^2\, d\mathbf{x} \tag{2.4}$$

Since $\Phi\,|_{\partial G} = 0$, *SE*-symmetrization is applicable to the function Φ. We also transform the non-uniformity function. We define the function $\nu_*(\mathbf{x})$ in the circle K as follows $\nu_*(\mathbf{x}) = \nu_*(r)$, which is equally measurable with the initial function $\nu(\mathbf{x})$ and does not increase as the radius r increases. Then by construction

$$\int_G \nu_0(\mathbf{x})\,|\nabla\Phi(\mathbf{x})|^2\, d\mathbf{x} = \int_K \nu_*(r)\,|\nabla SE(\Phi)(r)|^2\, d\mathbf{x} \tag{2.5}$$

By virtue of Theorem 1

$$\int_G \Phi(\mathbf{x})\, d\mathbf{x} \leqslant \int_G |\Phi(\mathbf{x})|\, d\mathbf{x} \leqslant \int_K SE(\Phi)(r)\, d\mathbf{x} \tag{2.6}$$

On the basis of (2.4)-(2.6) we have

$$C \leqslant 4\int_K SE(\Phi)(r)\, d\mathbf{x} - \int_K \nu_*(r)\,|\nabla SE(\Phi)(r)|^2\, d\mathbf{x} \leqslant$$
$$\sup_{\varphi,\ \varphi|_{\partial G}=0} \left[4\int_K \varphi\, dx - \int_K \nu_*(r)\,|\nabla\varphi(\mathbf{x})|^2\, d\mathbf{x}\right] = C_*$$

Here C_* is the stiffness under torsion of a circular rod with shear modulus $\mu_*(r) = 1/\nu_*(r)$. The function $\mu_*(r)$ is obviously codirectional with r, i.e., does not decrease as the radius increases.

We note that the estimate obtained is simultaneously also the solution of the corresponding optimization problem on the selection of the shape of the section of a twisted rod and the most logical distribution of the non-uniformity.

Longitudinal vibrations of an non-uniformity elastic rod. The equation and boundary conditions describing the longitudinal vibrations of an elastic rod with clamped ends and density $\rho(x)$, Young's modulus $E(x)$ and sectional area $F(x)$ variable over the length have the form /6/

$$\frac{d}{dx}\left[Q(x)\frac{dy}{dx}\right]+\lambda M(x)\,y(x)=0,\quad y(-l)=y(l)=0 \tag{2.7}$$
$$Q(x)=E(x)\,F(x),\quad M(x)=\rho(x)\,F(x)$$

where $2l$ is the rod length and λ is the square of the natural frequency. We call the function $Q(x)$ the generalized stiffness and $M(x)$ the generalized density of the non-uniform rod.

It is well-known /6/ that the square of the fundamental frequency (the minimal frequency) of the natural vibrations ω can be defined as follows:

$$\omega=\inf_{v(x),\ v(-l)=v(l)=0}\frac{\langle Q(x)\,[dv/dx]^2\rangle}{\langle M(x)\,v^2(x)\rangle} \tag{2.8}$$

Here and henceforth the angular brackets denote integration with respect to x in the segment $[-l, l]$.

Theorem 3. Among all the equally-measurable distribution functions of the generalized stiffness $Q(x)$ and density $M(x)$ for longitudinal vibrations of a non-uniform elastic rod with clamped ends, the minimum frequency of the fundamental of the natural vibrations corresponds to the case when $Q(x)$ and $M(x)$ are symmetrical about the middle of the rod and do not increase during motion from the middle to the ends of the rod.

Proof. Let $y(x)$ be the eigenfunction corresponding to the fundamental frequency for $Q(x)$ and $M(x)$

$$\omega=\langle Q(x)\,[dy/dx]^2\rangle/\langle M(x)\,y^2(x)\rangle \tag{2.9}$$

and let the functions $Q_0(x)$ and $Q(x), M_0(x)$ and $M(x)$ be equally-measurable, where $Q_0(x)$ is oppositely directed to $|dy/dx|$ and $M_0(x)$ is codirectional with $|y(x)|$. By virtue of the property mentioned above we have

$$\omega\geqslant q_0/m_0,\quad q_0=\langle Q_0(x)\,[dy/dx]^2\rangle,\quad m_0=\langle M_0(x)\,y^2(x)\rangle \tag{2.10}$$

We construct the functions $Q_*(x)$ and $M_*(x)$, which are equally measurable with the original functions $Q(x)$ and $M(x)$, respectively, symmetrical about the middle of the rod and non-increasing during motion from the middle to the ends of the rod. Then, by construction and by virtue of (1.9)

$$q_0=\langle Q_*(x)\,[dSE(y)(x)/dx]^2\rangle$$
$$m_0=\langle M_*[S(y)(x)]^2\rangle\leqslant\langle M_*(x)\,[SE(y)(x)]^2\rangle$$

Taking (2.10) into account, we finally obtain

$$\omega\geqslant\inf_{v(x),\ v(-l)=v(l)=0}\frac{\langle Q_*(x)\,[dv/dx]^2\rangle}{\langle M_*(x)\,v^2(x)\rangle}=\omega_* \tag{2.11}$$

where ω_* is the square of the fundamental frequency of the natural vibrations corresponding to the distribution of inhomogeneities of the form $Q_*(x), M_*(x)$.

We note that the eigenfunction $y_*(x)$ corresponding to the fundamental frequency $\sqrt{\omega_*}$, is invariant under *SE*-symmetrization. Indeed, in the opposite case it could yield a minimum of the functional (2.11) since application of the *SE*-symmetrization operation to it does not increase the value of the functional.

Theorem 3 generalizes the results of /2/ in which analogous inequalities are obtained in the case when only the density is inhomogeneous while Young's modulus and the sectional area are constant along the rod length.

Remark 3. The following theorem can be proved by a slight modification of the symmetrization operation introduced above.

Theorem 4. Let the surface S_1 bound a domain $G_1\subset R^n$ and let a surface S_2 bound a domain $G_2\subset R^n$, where $G_1\subset G_2$. In the domain $G_2\setminus G_1$ we consider the boundary value problem

$$\nabla\cdot(k(\mathbf{x})\,\nabla U)=0,\quad \mathbf{x}\in G_2\setminus G_1,\quad k(\mathbf{x})\geqslant 0 \tag{2.12}$$
$$U|_{S_1}=0,\quad U|_{S_2}=1$$

The quantity

$$I(k(\mathbf{x}), G_1, G_2) = \int\limits_{G_2 \setminus G_1} k(\mathbf{x}) |\nabla U_0(\mathbf{x})|^2 d\mathbf{x}$$

is investigated, where $U_0(\mathbf{x})$ is a solution of (2.12). The following assertion holds: among all the surfaces S_1, S_2 bounding the domains G_1, G_2 of a given volume, and among all the equally measurable functions $k(\mathbf{x})$, the minimum $I(k(\mathbf{x}), G_1, G_2)$ is reached in the case when G_1, G_2 are concentric spheres, and the function $k(x)$ is defined in a spherical layer $G_2 \setminus G_1$ which is spherically symmetric and does not decrease as the radius increases.

The boundary value problem (2.12) is encountered, say, in problems of a steady-state temperature of diffusion distribution for non-uniform heat conduction or permeability, respectively, of the medium. The quantity $I(k(\mathbf{x}), G_1, G_2)$ characterizes the heat or mass flow through the surface S_2.

Mathematically, Theorem 4 generalizes the isoperimetric inequality for the electrostatic capacitance /1/ corresponding to the case $k(\mathbf{x}) = \text{const}$.

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Translated by M.D.F.

PMM U.S.S.R.,Vol.52,No.3,pp.382-389,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

SOLID PHASE SEEDS IN A DEFORMABLE MATERIAL*

L.B. KUBLANOV and A.B. FREIDIN

An equilibrium solid phase see in a linearly elastic medium is considered. The problem of a medium with new phase equilibrium domains is reduced to equations of elasticity theory for an inhomogeneous medium with a special kind of definite "phase" deformation under an additional phase equilibrium condition /1/ that imposes a constraint on the shape of the phase boundary.

An ellipsoidal inclusion of an anisotropic phase is considered in an unbounded isotropic medium in a homogeneous external field of stress. It is proved that the tensor being defined by the phase deformation, by a change in the elastic moduli and stresses within the inclusion and having the meaning of a density tensor for dislocation moments indiced by a new phase domain, is global in the case of an equilibrium inclusion. The stress fields in an equilibrium two-phase configuration (TC) are determined by this characteristic property; the surface of the equilibrium ellipsoid turns out to be a surface of equal and constant principal values of the jump of the stress tensor and the constant principal value of the jump of the strain tensor. The stress perturbation tensor deviators within the

*Prikl.Matem.Mekhan.,52,3,493-501,1988